Theory of equations

1. Determine the number of real roots of the following equations by first finding the zeros of the derived polynomial, and locate them between consecutive integers.

(a) $x^4 + 4x^2 - 8x^3 - 1 = 0$

(b) $8x^5 - 5x^4 - 40x^3 - 50 = 0$

2. Determine the number of real roots of the following equations and locate each between consecutive integers.

(a) $x^4 + 2x^2 + 3x - 1 = 0$

(b) $x^5 - 2x^3 + x - 10 = 0$

3. α, β, γ are the roots of the equation $x^3 - px + q = 0$. Show that the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$, is $x^3 - 6px^2 + 9p^2x + (27q^2 - 4p^3) = 0$. Hence determine the condition that the roots of the first equation should be real.

4. Find the range of values of λ for which the equation $x^2 - 3x + 4 = \lambda(1 + 2x)$ has real roots. Illustrate your results by drawing the graphs of $y = x^2 - 3x + 4$ and $y = \lambda(1 + 2x)$ for different value of λ.

For what values of x is $3 + 6x$ greater than $(x^2 - 3x + 4)$?

5. By means of a graph, or otherwise, determine the values of t for which the equation $(x - 1)^2 (x - a) + t = 0$ has three real roots, where a is a given constant greater than 1.

Prove that, whatever the values of a, t, the roots α, β, γ are connected by the relation $\beta\gamma + \gamma\alpha + \alpha\beta - 2(\alpha + \beta + \gamma) + 3 = 0$.

6. Prove that the equation $ax^2 + bx + c = 0$ has no rational root if a, b, c are odd integers.

7. **(a)** The coefficients of the polynomial $P(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ are integers. Show that if p, q are both even (or odd) integers, then $P(p) - P(q)$ is even.

(b) Show that if $P(0)$ and $P(1)$ are both odd integers, the equation $P(x) = 0$ has no integral roots.

8. If a, b, c, A, B, C are rational and if the equation $ax^2 + bx + c = 0$ has an irrational root α, prove that $A\alpha^2 + B\alpha + C$ is rational if and only if $Ab = Ba$.

Obtain a necessary and sufficient condition for $A\alpha^3 + B\alpha^2 + C\alpha$ to be rational.

If $a_0, a_1, \ldots, a_n$ are rational, prove that $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ is rational if and only if the remainder on dividing $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ by $ax^2 + bx + c$ is independent of x.

9. Show by a calculus method that $x^3 - px + q$ will have one maximum and one minimum value if $p > 0$. If this condition is satisfied find the maximum and minimum values of the expression. Find the condition that the maximum value should be positive and the minimum value negative. Hence show that the cubic equation $x^3 - px + q = 0$ has three real roots if $4p^3 \geq 27q^2$, and only one real root if $4p^3 < 27q^2$.

State how many roots the following equations have :

(a) $x^3 - 2x + 7 = 0$

(b) $3x^3 + 4x - 2 = 0$

(c) $4x^3 - 7x + 3 = 0$.

10. **(a)** Transform the cubic equation $x^3 + Px^2 + Qx + R = 0$ into another equation lacking x^2 term .

(b) Show that by putting $x = y + z$, the equation $x^3 - 15x = 126$, becomes $y^3 + z^3 + (3yz - 15)x = 126$.

If further we choose $3yz - 15 = 0$, show that y^3, z^3 are the roots of the equation $t^2 - 126r + 125 = 0$. If y_0, z_0 are one admissible pair of values of y, z, show that the other admissible pairs are $\omega y_0, \omega^2 z_0$, where ω, ω^2 are the complex cube roots of unity. Hence solve the equation $x^3 - 15x = 126$.

11. **(a)** If α, β are the roots of the equation $x^2 + px + q = 0$, express $(\alpha^3 - \beta)(\beta^3 - \alpha)$ in terms of p and q. Hence or otherwise, show that if one of the roots of the equation $x^2 + px + q = 0$ is equal to the cube of the other, then $(p^2 - 2q)^2 = q(q + 1)^2$ and conversely .

(b) If α, β are the roots of the equation $x^2 + px + q = 0$, find the equation whose roots are $t\alpha + \beta$ and $t\beta + \alpha$. If p, q are real and $p^2 - 4q < 0$, $p \neq 0$, show that, for all real values of t, the roots of the new equation are different from zero .

12. **(a)** Find the values of p, q so that p^2 and q are the roots of $x^2 + bx + q = 0$.

(b) The equation $x^2 + bx + c = 0$ has the roots α, β. Find the equation whose roots are

$\dfrac{1}{1+\alpha^2}, \dfrac{1}{1+\beta^2}$. Solve the equation obtained when $b = 1$ and $c = -1$.

13. Let b, c be real numbers. The cubic equation $x^3 + 3x^2 + bx + c = 0$ has three distinct real roots which are in geometric progression . Show that there are unique values b and c such that the roots of this equation are integers, and find its equation and its roots.

14. Find the necessary and sufficient conditions on the coefficients of $x^4 + a_3 x^3 + a_2x^2 + a_1x + a_0 = 0$ which ensure that whenever z is a root so is $1/z$.

Hence show that the roots of a quartic equation of this type may be formed by solving several appropriate quadratic equations .

15. The quartic equation $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the cubic equation with roots $\alpha\beta + \gamma\delta$, $\beta\gamma + \delta\alpha$, $\gamma\alpha + \beta\delta$. Supposing that methods of solving quadratic and cubic equation are known, describe a procedure for solving quartic equation .

16. **(a)** Prove by induction on k that $x^k + x^{-k}$ can be expressed as a polynomial $p_k(z)$ in $z = x + x^{-1}$. Determine $p_2(z)$ and $p_3(z)$.

(b) If α is a root of the polynomial equation $x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0$, show that $\alpha + \alpha^{-1}$ is a root of the polynomial equation

$$z^3 + az^2 + (b-3)z + (c-2a) = 0$$

(c) Determine all roots of $x^4 + 4x^3 + 5x^2 + 4x + 1 = 0$.

17. The cubic equation $x^3 + 3qx + r = 0$ $(r \neq 0)$ has roots α, β, γ.

Verify that the sextic equation $r^2(x^2 + x + 1)^3 + 27q^3x^2(x+1)^2 = 0$ is satisfied by $\dfrac{\alpha}{\beta}$.

Comment on this result in relation to the roots of the cubic in the cases

(a) $q = 0$ and

(b) $4q^3 + r^2 = 0$ **.**

18. **(a)** State, without proof, the A.M. – G.M. inequality.

(b) The equation $f(x) \equiv x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ has n distinct positive roots.

Writing $a_i = (-1)^i \binom{n}{i} b_i^{\,i}$, prove that $b_{n-1} > b_n$.

By considering $f'(x)$, prove that $b_1 > b_2 > \ldots > b_{n-1} > b_n$.